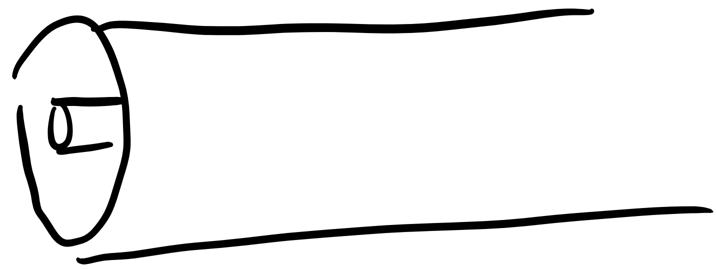
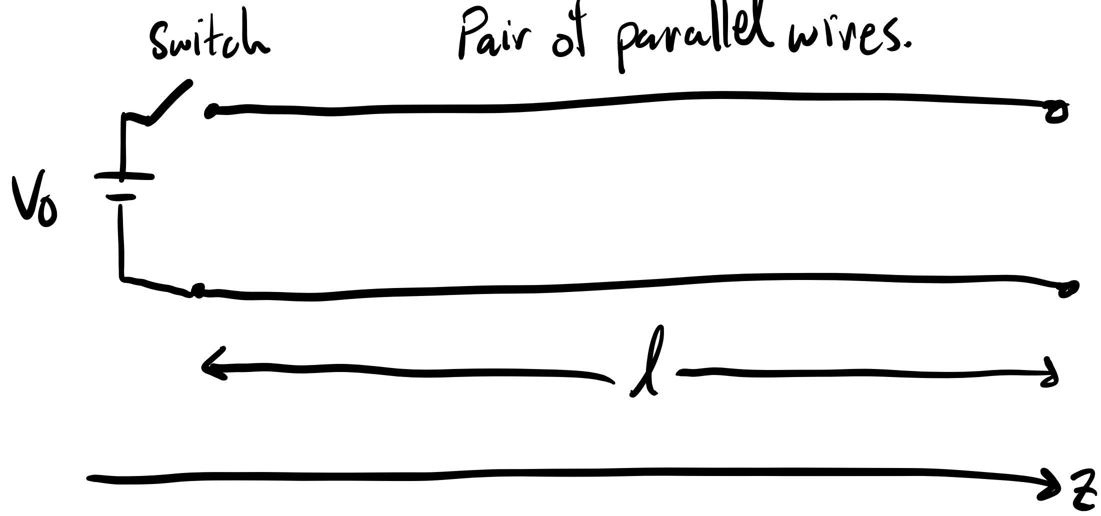
PHYS 331 - September 27, 2022

Transmission Lines.

Coaxial cable





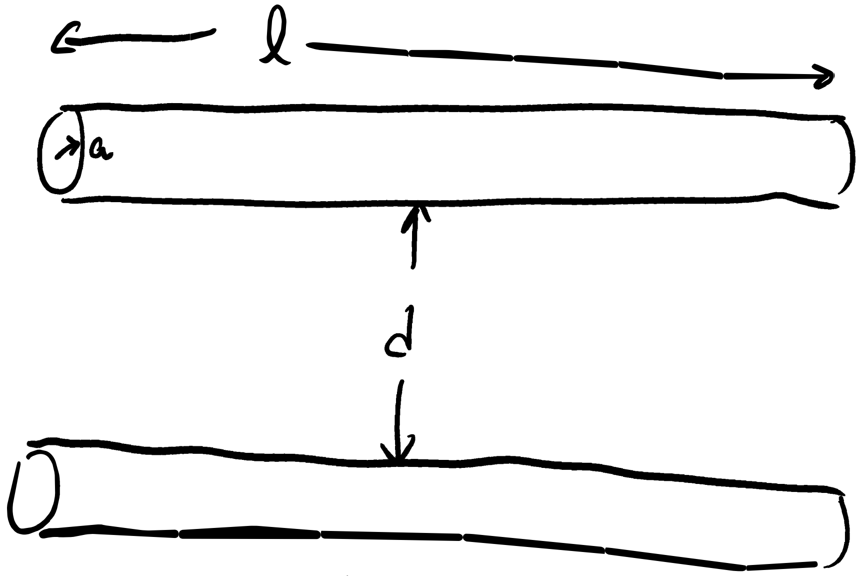
When switch is closed, get a pot. diff. across wires → wires store charge → capacitance

$$C = \frac{q}{\Delta V}$$

Must be a transient current to transport the charge ⇝ inductance

$$V_L = L\frac{dI}{dt}$$

$d \gg a$



$$\int \vec{E}\cdot d\vec{A} = \frac{q_{encl}}{\varepsilon_0}$$

$$\Delta V = -\int \vec{E}\cdot d\vec{\ell}$$

capacitance per unit length.

$$\frac{C}{\ell} = C_\ell = \frac{\pi \varepsilon_0}{\ln\left(\frac{d}{a}\right)}$$

$$\int \vec{B}\cdot d\vec{\ell} = \mu_0 i_{encl}$$

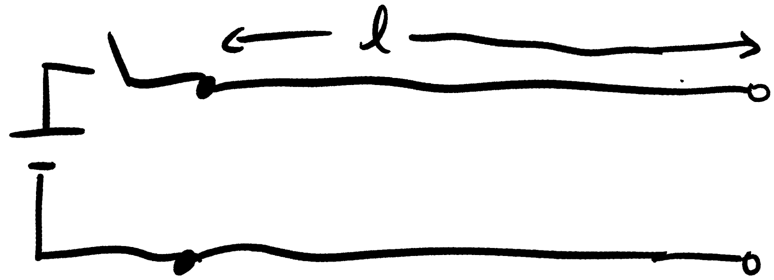
$$\Phi_B = \int \vec{B}\cdot d\vec{A}$$

$$V_L = -\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$$

inductance per unit length

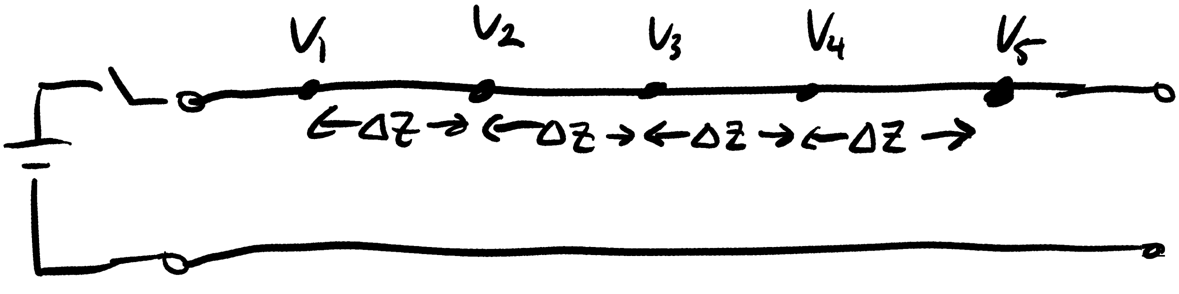
$$\frac{L}{\ell} = L_\ell = \frac{\mu_0}{\pi}\ln\left(\frac{d}{a}\right)$$

Model parallel wires as a circuit.



$$\Downarrow$$

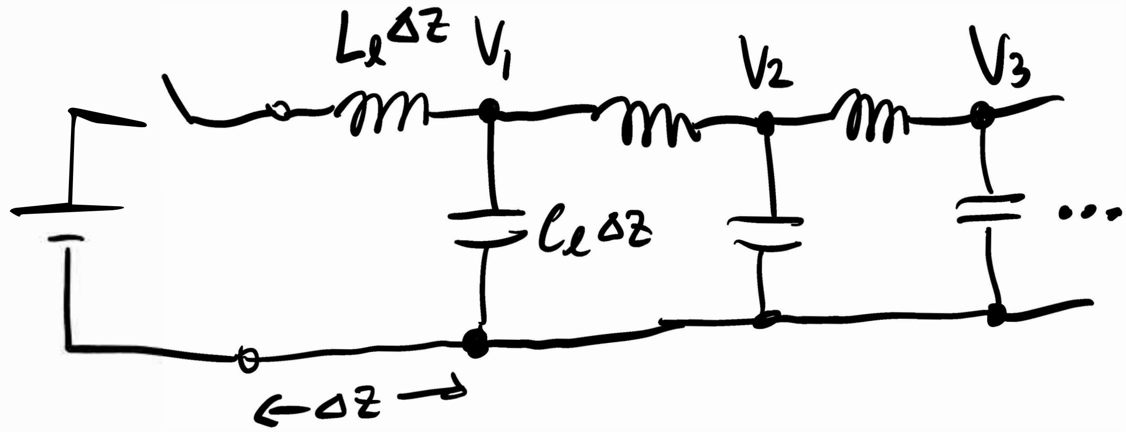
If wires are very long...



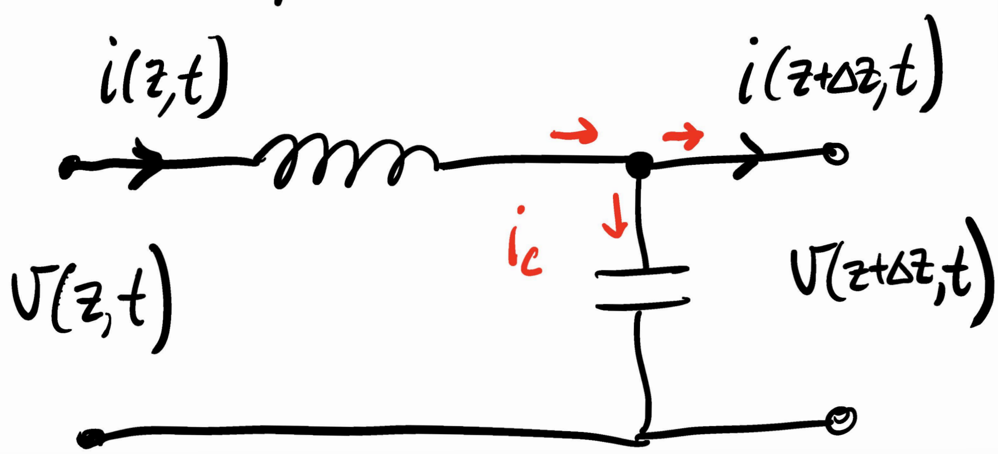
$V_1 > V_2 > V_3 > V_4 > V_5$ a short time after switch is closed.

The simple circuit model fails for long transmission lines.

Model long transmission line as a distributed LC network.



Consider a single branch of this circuit



Kirchhoff loop rule

$$v(z,t) - L_\ell \Delta z \frac{\partial i(z,t)}{\partial t} = v(z+\Delta z, t)$$

$$\therefore \quad \frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = -L_\ell \frac{\partial i(z,t)}{\partial t}$$

In $\Delta z \to 0$ limit, we have

$$\boxed{\frac{\partial v(z,t)}{\partial z} = -L_\ell \frac{\partial i(z,t)}{\partial t}} \quad ①$$

Kirchhoff Jcn Rule:

$$C = \frac{q}{\Delta V_c} \qquad \therefore q = C \Delta V_c$$

$$\therefore i_c = C \frac{dv_c}{dt}$$

$$i(z,t) = i_c + i(z+\Delta z, t)$$

Now sub $i_c = C_\ell \Delta z \dfrac{\partial v(z+\Delta z, t)}{\partial t}$

$$\therefore i(z,t) = C_\ell \Delta z \frac{\partial v(z+\Delta z, t)}{\partial t} + i(z+\Delta z, t)$$

$$\therefore \frac{i(z+\Delta z, t) - i(z,t)}{\Delta z} = -C_\ell \frac{\partial v(z+\cancel{\Delta z}, t)}{\partial t}$$

In limit $\Delta z \to 0$

$$\boxed{\frac{\partial i(z,t)}{\partial z} = -C_\ell \frac{\partial v(z,t)}{\partial t}}$$ ②

Assume that v & i are harmonic

$\rightarrow$ $v(z,t) = V(z)e^{j\omega t}$

$\rightarrow$ $i(z,t) = I(z)e^{j\omega t}$

From ①

$$\frac{\partial}{\partial z}\left(V(z)e^{j\omega t}\right) = -L_\ell \frac{\partial}{\partial t}\left(I(z)e^{j\omega t}\right)$$

$$\frac{\partial V(z)}{\partial z}\cancel{e^{j\omega t}} = -j\omega L_\ell I(z)\cancel{e^{j\omega t}}$$

$$\boxed{\frac{\partial V}{\partial z} = -j\omega L_\ell I} \Rightarrow \frac{\partial^2 V}{\partial z^2} = -j\omega L_\ell \frac{\partial I}{\partial z}$$

From (2)

$$\frac{\partial}{\partial z}\left(I(z)e^{j\omega t}\right) = -C_\ell \frac{\partial}{\partial t}\left(V(z)e^{j\omega t}\right)$$

$$\frac{\partial I}{\partial z}\cancel{e^{j\omega t}} = -j\omega C_\ell \, V(z)\cancel{e^{j\omega t}}$$

$$\boxed{\frac{\partial I}{\partial z} = -j\omega C_\ell V} \Rightarrow \frac{\partial^2 I}{\partial z^2} = -j\omega C_\ell \frac{\partial V}{\partial z}$$

$$\therefore \frac{\partial^2 V}{\partial z^2} = -j\omega L_\ell\left(-j\omega C_\ell V\right)$$

$$\boxed{\therefore \frac{\partial^2 V}{\partial z^2} = -\omega^2 L_\ell C_\ell V}$$

The wave Eq'ns.

Likewise:

$$\frac{\partial^2 I}{\partial z^2} = -j\omega C_\ell \left(-j\omega L_\ell I\right)$$

$$\boxed{\frac{\partial^2 I}{\partial z^2} = -\omega^2 L_\ell C_\ell I}$$

General sol'n to voltage eq'n:

$$V(z) = V_+ e^{-j\beta z} + V_- e^{j\beta z}$$

check:

$$\boxed{\frac{\partial V}{\partial z} = -j\beta V_+ e^{-j\beta z} + j\beta V_- e^{j\beta z}}$$

$$\frac{\partial^2 V}{\partial z^2} = -\beta^2 V_+ e^{-j\beta z} - \beta^2 V_- e^{j\beta z}$$

$$= -\beta^2 \underbrace{\left(V_+ e^{-j\beta z} + V_- e^{j\beta z}\right)}_{V(z)} = -\beta^2 V(z)$$

Assumed sol'n for $V(z)$ is valid provided

$$\beta^2 = \omega^2 L_\ell C_\ell$$

$$\boxed{\beta = \omega\sqrt{L_\ell C_\ell}}$$

Full time-dependent voltage is:

$$v(z,t) = V(z)e^{j\omega t}$$

$$= \left(V_+ e^{-j\beta z} + V_- e^{j\beta z}\right)e^{j\omega t}$$

$$= V_+ e^{j(\omega t - \beta z)} + V_- e^{j(\omega t + \beta z)}$$